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A new approach for automatic control modeling, analysis and design in fully fuzzy environment

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Fuzzy controllability and observability;
Fuzzy inverted pendulum system

Abstract The paper presents a new approach for the modeling, analysis and design of automatic control systems in fully fuzzy environment based on the normalized fuzzy matrices. The approach is also suitable for determining the propagation of fuzziness in automatic control and dynamical systems where all system coefficients are expressed as fuzzy parameters. A new ***consolidity chart*** is suggested based on the recently newly developed ***system consolidity index*** for testing the ***susceptibility*** of the system to withstand changes due to any system or input parameters changes effects. Implementation procedures are elaborated for the consolidity analysis of existing control systems and the design of new ones, including systems comparisons based on such implementation consolidity results. Application of the proposed methodology is demonstrated through illustrative examples, covering fuzzy impulse response of systems, fuzzy Routh–Hurwitz stability criteria, fuzzy controllability and observability. Moreover, the use of the consolidity chart for the appropriate design of control system is elaborated through handling the stabilization of inverted pendulum through pole placement technique. It is also shown that the regions comparison in consolidity chart is based on type of consolidity region shape such as elliptical or circular, slope or angle in degrees of the centerline of the geometric shape, the centroid of the geometric shape, area of the geometric shape, length of principal diagonals of the shape, and the diversity ratio of consolidity points for each region. Finally, it is recommended that the proposed consolidity chart approach be extended as a unified theory for modeling, analysis and design of continuous and digital automatic control systems operating in fully fuzzy environment.

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1. Introduction

The majority of the applications of fuzzy theory to automatic control systems are basically directed toward the development of fuzzy logic controllers (FLCs) for linear and nonlinear systems with given or unknown systems' models [1–5]. A wide class of these controllers constitutes several components namely the rule-base engine, the fuzzification process, the inference mechanism and the defuzzification process. To avoid

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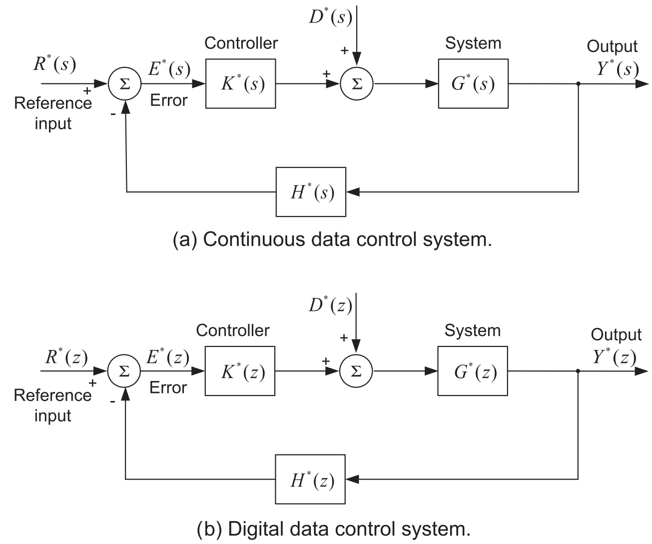
defuzzification ambiguities which may arise from more than one crisp output value, some weighted-based techniques are commonly used such as the averaging, the center of gravity (centroid), or the root-sum-square methods [6–9]. Other FLCs are based on the conventional fuzzy control (Mamdani Type fuzzy control), fuzzy PID control, neuro-fuzzy control, fuzzy sliding-mode control, adaptive fuzzy control, supervisory fuzzy control, and the Takagi and Sugeno (T–S) model-based fuzzy control [10–15].

An important aspect that goes along with the development of FLCs is the solution of the general modeling and analysis aspect of automatic control systems operating in fully fuzzy environment. This is the general case where all inputs and system parameters are fuzzy variables. Two typical examples of linear control systems for the ***continuous*** and ***digital*** data cases are shown in Fig. 1. There is a definite need to extend the development of all well proven techniques and methodologies of the deterministic case to the full fuzziness situation. The extension should allow the systematic calculations of propagated fuzziness inside the control system of different configurations and representations.

The modeling and analysis of automatic control systems operating in fully fuzzy environment is not effectively solved in the literature [16–18]. There are many approaches that are carried out to handle this problem using the conventional fuzzy theory. These approaches suffer many drawbacks such as the processing of the solution becomes very cumbersome with the increase of system dimensionality. Moreover, the results obtained by such like approaches are not linear and thus not reversible, leading to that the results obtained in the *forward path* will be different than the *backward path* [19,20]. Other techniques, using the direct implementation of fuzzy matrices, also have many shortcomings [21–23]. The main hindrance of their spread is heavily related to their impracticability of their present operations (Max, Min, Max.Min, and Min.Max) as they do not reflect any corresponding real-life physical meanings and cause irreversibility and nonlinearity in their processing.

Gabr and Dorrah presented their new notion of Arithmetic and Visual fuzzy logic-based representations [24–30]. The approach was based on the normalized fuzzy matrices, where every parameter is expressed by its value and corresponding fuzzy level. It is shown by Gabr [29,30] that the proposed Arithmetic fuzzy logic-based representation has corresponding mathematical functions with the conventional fuzzy theory. However, the new approach provides a much easier *arithmetic* rather than *logic* calculations forum that makes its application much practical and effective. Reported methodological experimentations and case studies applications of the Arithmetic and Visual fuzzy logic-based representations were successful in solving some *preliminary* classes of fuzzy global optimization and operations research techniques [24–28].

Such fuzziness approach has resulted in the appearance of a new system index named as "***system consolidity index***".[1] ***Consolidity*** (the *act* and *quality* of consolidation) is measured by the systems output reactions versus combined input/system parameters reaction when subjected to varying environments and events [31–36]. Moreover, consolidity can govern the ability of systems to withstand changes when subjected to incurring events or varying environments "***on and above***" normal operation during the system change pathway.



( )* means fuzzy parameter.

Fig. 1 Two examples of ***continuous*** and ***digital*** data control systems operating in fuzzy environments.

Though the topic of ***consolidity theory*** has commenced recently in year 2010, there are some good applications of such theory for the analysis and design of automatic control theory. In such applications, the opposite relationship between consolidity and both stability and controllability of state space representation systems was investigated in [33]. Moreover, several examples of applications to automatic control systems were carried out such as the fuzzy design of inverted pendulum using pole replacement method, the optimal design of the fuzzy linear quadratic regulator problem, and the fuzzy Lyapunov stability analysis of the drug concentration control problem [35]. In all these applications, the overall values of consolidity index (average of calculated consolidity points) are only considered in the study without going into any further investigations of the geometric distributions or the diversity analysis of the various consolidity points.

In the following section, the implementation of the consolidity theory is elaborated for further use in the analysis and design of control systems.

2. Methodology development

2.1. Description of the operating fuzzy environments

The fuzzy environments used in system consolidity analysis could be classified as shown in Table 1, and their representations are elucidated in Fig. 2.

In order not to complicate the mater unduly in this paper, our analysis will be based on classes E_O and E_B. The system consolidity analysis applies similarly to other fuzzy environment classes.

[1] Consolidity could be regarded as a general internal property of physical systems that can also be defined far from fuzzy logic or rough sets. Other consolidity indices, however, could be defined by researchers but the concept will still remain the same.

Table 1 Various classifications of fuzzy environments for consolidity analysis.

Ser.	Class name	Class abbreviation	Description
1	Open fully fuzzy environment	E_O	It is open fully fuzzy environment where all fuzzy levels can equally change all over the positive and negative values of the environment
2	Conditionally open fully fuzzy environment	E_C	It is an open fully fuzzy environment but it has the following conditions: (i) The changes of fuzzy levels of parameters may be correlated, and/or (ii) The fuzzy levels may have certain possibility of occurrence similar to Type-2 Fuzzy algebra.
3	Bounded fully fuzzy environment	E_B	It is a restricted fully fuzzy environment where all fuzzy levels can equally change all over the positive and negative within restricted ranges in the environment
4	Partial fuzzy environment	E_P	It is a restricted partial fuzzy environment where only some fuzzy (or none fuzzy) levels can differently or conditionally change within restricted ranges in the partial environment

2.2. The consolidity methodology

The consolidity methodology for the analysis and design of control systems is based in modeling the system input and parameters as fuzzy variables, leading to a corresponding output of the similar fuzzy nature. A system operating at a certain stable original state in fully fuzzy environment is said to be ***consolidated*** if its overall output is suppressed corresponding to their combined input and parameters effect, and vice versa for ***unconsolidated*** systems. Neutrally consolidated systems correspond to marginal or balanced reaction of output, versus combined input and system.

In general, the output fuzziness behavior toward input fuzziness could differ from one system to another. Examples of these behaviors are as follows:

(i) Outputs could absorb the input fuzziness and give smaller or diminishing output fuzziness.
(ii) Outputs could yield almost the same level of input fuzziness.
(iii) Output could give higher output fuzziness compared to input fuzziness.

Let us assume a general system operating in fully fuzzy environment, having the following elements:

Input parameters:

$$\underline{I} = (V_{I_i}, \ell_{I_i}) \tag{1}$$

such that $V_{I_i},\ i = 1, 2, \ldots, m$ describe the deterministic value of input component I_i, and ℓ_{I_i} indicates its corresponding fuzzy level.

System parameters:

$$\underline{S} = (V_{S_j}, \ell_{S_j}) \tag{2}$$

such that $V_{S_j},\ j = 1, 2, \ldots, n$ denote the deterministic value of system parameter S_j, and ℓ_{S_j} denotes its corresponding fuzzy level.

Output parameters:

$$\underline{O} = (V_{O_i}, \ell_{O_i}) \tag{3}$$

such that $V_{O_i},\ i = 1, 2, \ldots, k$ designate the deterministic value of output component O_i, and ℓ_{O_i} designates its corresponding fuzzy level.

We will apply in this investigation, the overall fuzzy levels notion, first for the combined input and system parameters, and second for output parameters. As the relation between combined input and system with output is close to (or of the like type) of the multiplicative relations, the multiplication fuzziness property is applied for combining the fuzziness of input and system parameters.

For the combined input and system parameters, we have the weighted fuzzy level to be denoted as the combined ***Input and System Fuzziness Factor*** F_{I+S}, given as:

$$F_{I+S} = \frac{\sum_{i=1}^{m} V_{I_i} \cdot \ell_{I_i}}{\sum_{i=1}^{m} V_{I_i}} + \frac{\sum_{j=1}^{n} V_{S_j} \cdot \ell_{S_j}}{\sum_{j=1}^{n} V_{S_j}} \tag{4}$$

Similarly, for the ***Output Fuzziness Factor*** F_O, we have

$$F_O = \frac{\sum_{i=1}^{k} V_{O_i} \cdot \ell_{O_i}}{\sum_{i=1}^{k} V_{O_i}} \tag{5}$$

Let the positive ratio $|F_O/F_{I+S}|$ defines the ***SystemConsolidity Index***, to be denoted as $F_{O/(I+S)}$. Based on $F_{O/(I+S)}$ the system consolidity state can then be classified as [20–33]:

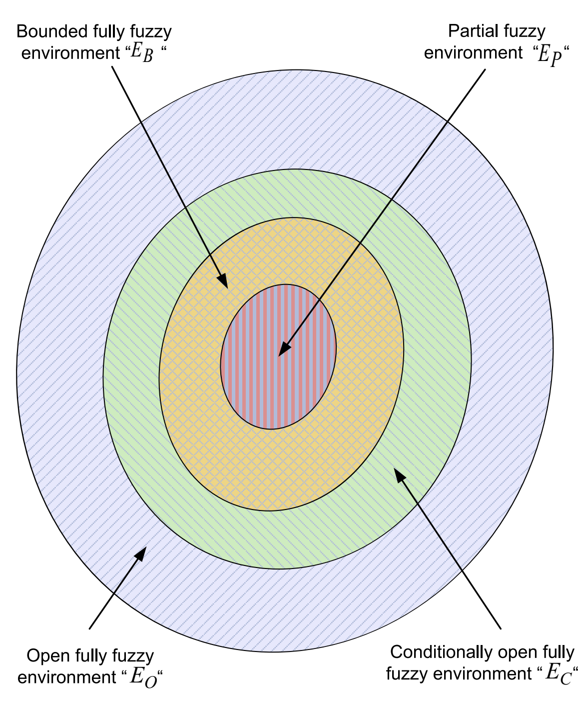


Fig. 2 Various classifications of different fuzzy environments.

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(i) ***Consolidated*** *if* $F_{O/(I+S)} < 1$, to be referred to as "*Class C*".
(ii) ***Neutrally Consolidated*** *if* $F_{O/(I+S)} \approx 1$, to be denoted by "*Class N*".
(iii) ***Unconsolidated*** *if* $F_{O/(I+S)} > 1$, to be referred to as "*Class U*".

For cases where the system consolidity indices lie at both consolidated and unconsolidated parts, the system consolidity will be designated as a mixed class or "*Class M*".

The selection of the fuzzy levels testing scenarios for both the system and input should follow the same usual consideration. First of all the input and system fuzzy values for system consolidity testing are selected as integer values to be preferably in the range ± 8 for open fuzzy environment and in the range ± 4 for bounded fuzzy environments. Nevertheless, the output fuzzy level could assume open values beyond these ranges based on the overall consolidity of the system. However, all over implementation procedure in the paper, the exact fraction values of fuzzy levels are preserved during the calculations and are rounded as integer values only at the final results [34].

It is remarked that the typical ranges of the consolidity indices $F_{O/(I+S)}$ based on previous real-life applications are as follows: ***very low*** (< 0.5), ***low*** (0.5–1.5), ***moderate*** (1.5–5), ***high*** (5–15), and ***very high*** (> 15) [34,35]. A good practical consolidated system should have the value of consolidity index $F_{O/(I+S)} \leqslant 1.5$.

2.3. The consolidity chart

The concept of implementing the consolidity theory to the analysis and design of control system is to plot for each system its consolidity chart defined as the relation between the ***Output Fuzziness Factor*** $|F_O|$ in the vertical axis (y) and the ***Input and System Fuzziness Factor*** $|F_{I+S}|$ in the horizontal axis (x).

The best way for sketching each system's region in the consolidity chart is to calculate representative points of ***output fuzziness factor*** (y-axis) versus ***input and the system fuzziness factor*** (x-axis) and plot all these x–y points first in the chart. The average consolidity index is then calculated based on these points and its value will represent the slope of the center line of the region under study. The boundary of the region can then be sketched around this center line embodying all (or the majority) of these fuzziness x–y points.

Examples of the consolidity regions or patterns of various consolidity classes are summarized in Table 2 and sketched in Fig. 3. The shapes of each region are assumed for simplicity of the elliptical. However, other geometric shapes such as the circular one could take place for various applications.

Based on these consolidity patterns, the degree of susceptibility of the system to withstand the effect of changes in system and input parameters can be evaluated. In fact, ***Consolidity index*** is an important factor in scaling system parameter changes when subjected to events or varying environment. For instance, for all coming events at say event state μ which are "***on and above***" the system normal situation or stand will lead to consecutive changes of parameters. Such changes follow the general relationship at any event step μ as: Δ Parameter change $^{\mu}$ = ***Function*** [consolidity $^{\mu}$, varying environment or event $^{\mu}$] [37,38]. Two important common cases in real life of such formulation are the ***linear*** (or linearized) and the ***exponential*** relationship**.**

2.4. Implementation approach of consolidity theory to control systems

2.4.1. Implementation approach for existing control systems

For the existing *man-made* or *natural* systems, the situation could be complicated. The testing of these existing systems could reveal the poor consolidity of such system. This is quite expected as we previously used to build all existing systems without considering the new concept of system consolidity. For existing *man-made* systems, the situation could be possible by altering parameters of the system within the utmost extent permitted for changes. As for *natural* systems, the system consolidity improvement matter could also be possible by interfering within the system parameters together with environment and trying to direct the physical process toward better targeted consolidity.

For ***existing automatic control systems***, they could be *firstly* examined for their consolidity behavior. Based on the obtained results, appropriate interventions are carried out for adjusting one or more of system key parameters or *possibly* controlling existing operating environment to attain improved consolidity without jeopardizing their stability or performance. This interfering approach for existing systems is illustrated in Fig. 4(a).

2.4.2. Implementation approach for new control systems design

For ***new automatic control systems*** design, the implementation of consolidity theory is much simpler. The designers commence using the conventional automatic control techniques leaving at the end one or more flexible (or changeable) parameters that consecutively be adjusted for preserving good system consolidity behavior. Several designs could be developed and then ranked within the framework of consolidity for selecting the best choices that also fulfill acceptable degrees of functionality.

The approach is also applicable for higher dimensional automatic control as it is based mainly on matrix formulations. For design analysis, conventional techniques are first used leaving one or two parameters of flexible ranges. The suggested consolidity technique can then be drawn by varying these parameters to obtain an improved design from the consolidity point of view.

In general, as the generation of these prototypes during the design process is not completely exhaustive. The terms of *superior* or *inferior* of consolidation remain as relative comparison. Such comparison is sufficient for all real-life applications as the system designers could follow later other cycles of improvement to locate a new better superior system that surmounts the old superior design. Such building approach for new systems is elucidated in Fig. 4(b).

It is remarked that during implementation each problem parameters should be addressed first as *symbols* and not be substituted (with fuzziness defined as their pairs or shadows). Conventional mathematics is then applied to the basic variables while the appropriate fuzzy algebra is implemented on their corresponding pairs or shadows (fuzziness). Parameters substitutions are made at the end step of solution leading to the calculation of the consolidity factors as specified by the problem analyst.

Table 2 Various classifications of system consolidity.

Ser.	Class name	Class abbreviation	Description
1	Consolidated	C	All values of Consolidity Index are less than 1, that is $F_{O/(I+S)} < 1$
2	Quasi-consolidated	$\tilde{C}$	A mixed system that is clearly inclined more toward consolidation such as the center of gravity (averaged value) has $F_{O/(I+S)} < 1$
3	Neutrally consolidated	N	All values of Consolidity Index are nearly 1, that is $F_{O/(I+S)} \approx 1$
4	Mixed class	M	All values of Consolidity Indices lie at both Consolidated and Unconsolidated zones, $F_{O/(I+S)} <$ and > 1
5	Quasi-unconsolidated	$\tilde{U}$	A mixed system that is clearly inclined more toward un-consolidation such as the center of gravity (averaged value) has $F_{O/(I+S)} > 1$
6	Unconsolidated	U	All values of Consolidity Index are more than 1, that is $F_{O/(I+S)} > 1$

For complicated symbolic manipulations (and computations) the use of Matlab Symbolic Toolbox, Mathematica or similar like software libraries could be highly effective to foster the consolidity theory through conducting its necessary derivations. This will enable the implementation of the consolidity analysis to wider classes of linear, nonlinear, multivariable and dynamic problems with different types of complexities.

2.5. Systems comparison based on the consolidity indices implementation results

The first step in the design of any specific problem is to carry out the consolidity indices of all various available scenarios. These results are extracted from only one overall consolidity index based on the design philosophy to be followed. Examples of such design basis are given as

(i) ***Average consolidity scores***: In this case, the average of all scored consolidity indices for each scenario is calculated and used solely for the selection of the most appropriate consolidated design.
(ii) ***Weighted average consolidity scores***: For the situations where the possibility of the input and system fuzziness is known, the weighted average values of the consolidity by these given possibilities are calculated and the results are used in the selection of design.
(iii) ***Worst consolidity scores***: In this case, the worst score of the consolidity index is chosen as an overall evaluation index. This could be the maximum (or minimum) of all scored indices if we are seeking the superior (or inferior) consolidated design

Another in-depth direction of comparisons is through plotting regions (patterns) of fuzziness behavior in the sketched consolidity chart similar to the ones shown in Fig. 3. It could appear from such figure that each consolidity region changes from system to another and follows certain geometric shapes such as the elliptical, circulars, or others. The geometric features of each consolidity region can be characterized by the following features:

Symbol	Description
R	Type of consolidity region shape such as elliptical, circular, or other shapes
S	Slope or angle in degrees of the centerline of the geometric shapes of the overall consolidity index S(degrees) = $\tan^{-1}$(overall consolidity index)
$C = (x, y)$	The centroid of the geometric shape expressed by its horizontal coordinate per unit (pu) and the vertical coordinate per unit
A	Area of the geometric shape of R in pu^2
l_1	Length of geometric (major) diagonal in direction of slope of the consolidity region (pu)
l_2	Length of geometric (minor) diagonal in perpendicular to the slope S of the consolidity region (pu)
l_2/l_1	Diversity ratio of consolidity points (unitless)

The comparison or ranking of each consolidity region will be based on less slope, less area and less diversity ratio (l_2/l_1). Moreover, the position of the centroid $C = (x, y)$ (upward or downward) within the geometric shape main centerline depends mainly on the nature of the affected input influences which are particular for each specific application. Higher values of such centers mean higher fuzzy input effects or influences. The above shown features of the consolidity charts will be the basis of the analysis of the various applications given in the following sections of the paper.

3. Methodology implementation to automatic control systems modeling

Consider the general differential equation [16]

$$\frac{d^n x}{dt^n} + a_{n-1} \cdot \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \cdot \frac{dx}{dt} + a_0 \cdot x = b_{n-1} \cdot \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_1 \cdot \frac{du}{dt} + b_0 \cdot u \ldots \quad (6)$$

where all the equation parameters are fuzzy numbers. These fuzzy numbers are expressed by their deterministic values and corresponding fuzzy level as described by the Arithmetic fuzzy logic-based representation.

Define a set of state variables for a typical fuzzy control system as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= -a_0 \cdot x_1 - \cdots - a_{n-1} \cdot x_n + u \end{aligned} \quad (7)$$

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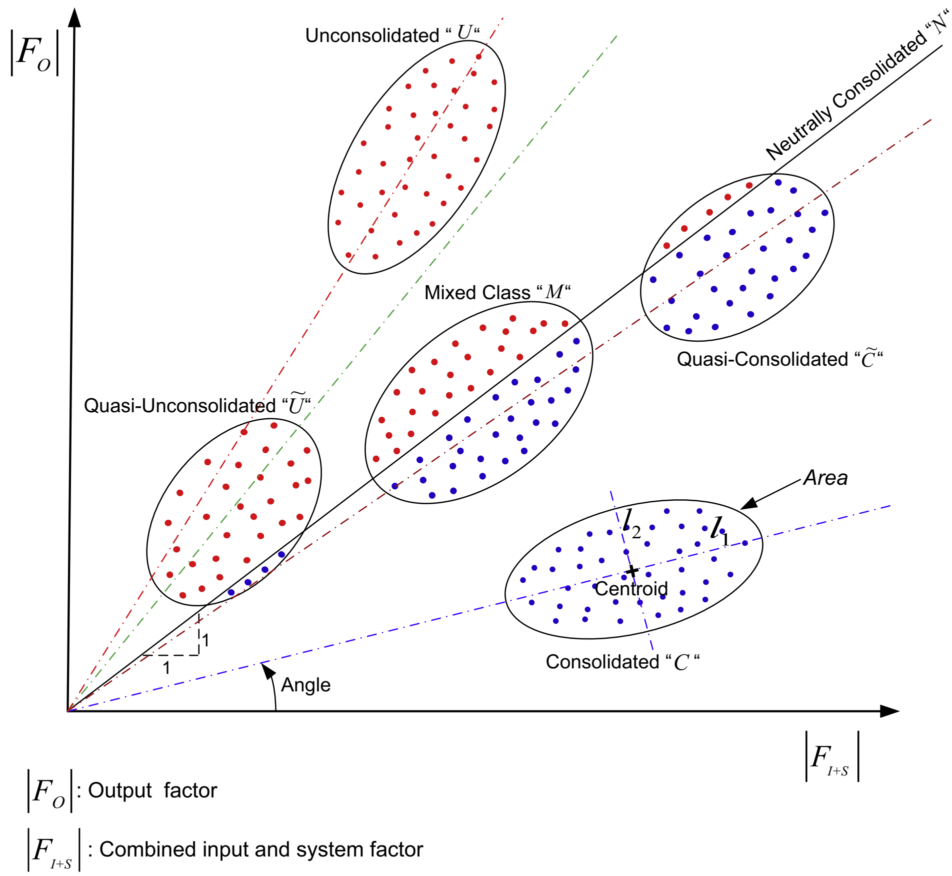

Fig. 3 The ***consolidity chart*** showing six different classes of system consolidity patterns (regions) as described in Table 2.

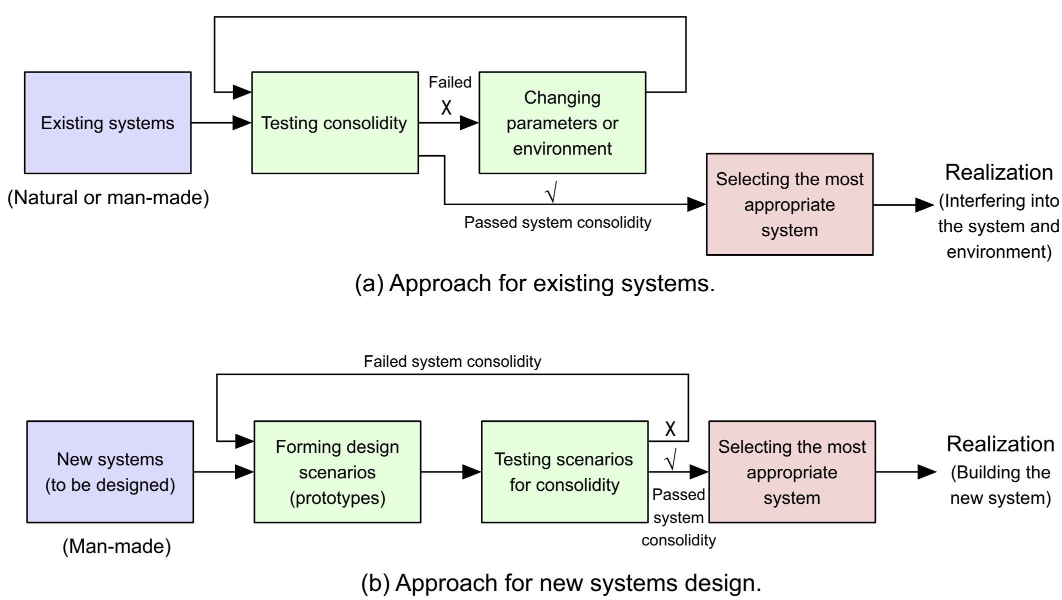

Fig. 4 Implementation approach of consolidity theory for both existing and new systems under design.

and an output equation

$$y = b'_0 \cdot x_1 + b'_1 \cdot x_2 + \cdots + b'_{n-1} \cdot x_n \quad (8)$$

where $b'_0 \quad b'_1 \quad b'_2 \quad \cdots \quad b'_{n-1}$ are fuzzy coefficients.

Then, the state equation is expressed as

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad (9)$$

The state-space representation of (9) is denoted as the controllable canonical form. The output equation is

$$y = \begin{bmatrix} b'_0 & b'_1 & b'_2 & \cdots & b'_{n-1} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad (10)$$

Consider now the state vector differential equation

$$\dot{x} = A \cdot x + B \cdot u \quad (11)$$

Taking Laplace transforms of (11), we get

$$sX(s) - x(0) = A \cdot X(s) + B \cdot U(s) \quad (12)$$

or equivalently

$$(s \cdot I - A) \cdot X(s) = x(0) + B \cdot U(s) \quad (13)$$

Using a state variable representation of a system, the characteristic equation is given by

$$|(s \cdot I - A)| = 0 \quad (14)$$

This yields the characteristics (closed-loop form) equation [16]:

$$a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \cdots + a_1 \cdot s + a_0 = 0 \quad (15)$$

The general form of the above system can be expressed in the form of system transfer function as

$$\frac{C}{R}(s) = \frac{G(s)}{1 + G(s) \cdot H(s)} = \frac{K_c \cdot (s - z_{cl}) \cdot (s - z_{c2}) \ldots (s - z_{cn})}{(s - p_{c1}) \cdot (s - p_{c2}) \ldots (s - p_{cn})} \quad (16)$$

where $s = p_{c1}, p_{c2}, \ldots, p_{cn}$ are closed-loop fuzzy poles, since their values make (16) infinite (also the roots of the characteristic equation) and $s = z_{c1}, z_{c2}, \ldots, z_{cn}$ are closed-loop fuzzy zeros, since their corresponding values of (13) are zero.

We present now the handling of the general form of fuzzy control system modeling and analysis using representative examples of the fourth-order systems.

4. Methodology implementation to control systems fuzzy impulse response

We demonstrate in this section how a fourth order system of the transfer function as expressed by (16) can be handled in a fully fuzzy environment where all the system coefficients are expressed in the Arithmetic fuzzy logic-based representation form. Let us introduce this example that describes the fuzzy response of a high-order control system operating in fully fuzzy environment. We introduce the example in a general form of fourth-order open-loop transfer function, as follows [16]:

$$X_0(s) = \frac{a_0}{s \cdot (s + b_1) \cdot (s^2 + c_1 \cdot s + c_2)} \quad (17)$$

where a_0, b_1, c_1 and c_2 are fuzzy parameters.

Eq. (17) may be written using partial fraction representation as

$$X_0(s) = \frac{A}{s} + \frac{B}{s + b_1} + \frac{C \cdot s + D}{(s + c_1/2)^2 + c_3^2} \quad (18)$$

where $c_3 = \sqrt{c_2 - c_1^2/4}, A, B, C, D,$ and c_3 are fuzzy coefficients. Equating coefficient of (17), we get

$$\begin{aligned} (s^3): &\quad 0 = A + B + C \\ (s^2): &\quad 0 = A \cdot (b_1 + c_1) + B \cdot c_1 + C \cdot b_1 \\ (s^1): &\quad 0 = A \cdot (c_2 + b_1 \cdot c_1) + B \cdot c_2 + D \cdot b_1 \\ (s^0): &\quad a_0 = A \cdot b_1 \cdot c_2 \end{aligned} \quad (19)$$

Using the Gaussian Elimination technique, the matrix equation of (19) can be solved with its corresponding fuzzy levels.

Illustrative example 1. As a numerical example, we choose the value of fuzzy parameters as shown in Table 3. The results of parameters $A, B, C,$ and$D$ are also shown in the same table. Accordingly, the inverse Laplace transform of (19) can be expressed as

$$X_0(t) = A - B \cdot e^{-b_1 \cdot t} - C \cdot e^{-c_1 \cdot t/2}[(c_4)^2 \cdot \sin c_3 \cdot t - \cos c_3 \cdot t] \quad (20)$$

such that $C_4 = \left(\frac{D}{C} - \frac{c_1}{2}\right)$ where $A, B, C, D, b_1, c_1, c_3,$ and C_4 are fuzzy parameters.

The consolidity pattern of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{I+S}|$ is shown in Fig. 5. The impulse response output solution pattern reveals slight unconsolidated distribution of the results, indicating relatively ***low susceptibility*** of the optimal solution for change versus any system and input parameters changes effect. Based on consolidity chart of Fig. 5, it can be seen that the control system is almost ***consolidated*** of class "*C*".

For the selected first four scenarios shown in Table 3, the fuzzy levels of impulse responses are given also in Table 3 and Fig. 6. The equations were solved in Excel sheet with built-in functions programmed using Visual Basic Applications (VBAs). In the implementation procedure, the exact values of fuzzy levels are preserved all over the calculations and are rounded to integer values only at the final result. It follows from the sketches of the impulse time response of Fig. 6 and Table 4 that the fuzziness is related to the time instant. The color of the response is an indication of the fuzzy level using the color coding shown in Table 5. Such colors are selected arbitrarily without restricting that corresponding positive and negative colors are conjugates (summation is either white or black). This is equivalent to the Visual fuzzy logic-based representation [24–27].

The analysis of the ***consolidity chart*** of the impulse response problem of Illustrative example 1 shown in Fig. 5 can be summarized as follows:

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Table 3 Consolidity analysis of the fuzzy impulse response of Illustrative example 1.

Type	Parameter	Value	Fuzzy level scenarios								
			I	II	III	IV	V	VI	VII	VIII	IX
Input	a_0	12.5	3	1	−1	−3	1	4	4	3	5
	b_1	0.5	−3	−1	1	3	6	4	5	6	4
	c_1	1.0	−2	−2	2	2	4	4	−4	2	4
	c_2	25.0	3	1	−1	−3	3	6	3	5	6
Output	A	1.000	3	1	−1	−3	−8	−6	−4	−8	−5
	B	−1.010	3	1	−1	−3	−8	−6	−4	−8	−5
	C	0.010	−4	−4	4	4	−3	−4	−15	−9	−3
	D	−0.495	−6	0	0	6	−2	−2	1	−2	−1
Consolidity index value $F_{O/(I+S)}$ [a]			0.0185	0.0802	0.0802	0.0185	0.8298	0.3825	0.4096	0.4444	0.1823

[a] Average value of consolidity index $F_{O/(I+S)} = 0.2718$.

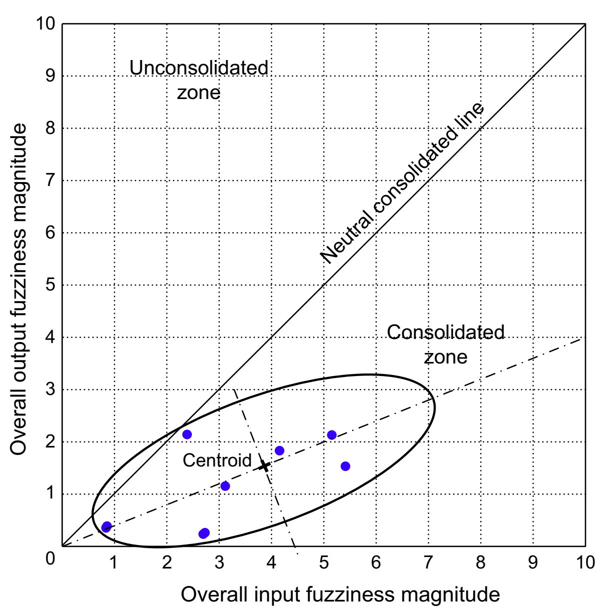

Fig. 5 Consolidity chart of the impulse response problem of Illustrative example 1.

Symbol	Meaning	Results
R	Shape type	Elliptical
S	Slope	15.21°, or $\tan^{-1}(0.27180)$
$C = (x, y)$	Centroid	(3.8481, 1.6203)
A	Area of shape	13.7414 pu^2
l_1	Length of major diagonal	6.9873 pu
l_2	Length of minor diagonal	2.5316 pu
l_2/l_1	Diversity ratio	0.3623

The results indicate that the consolidity region has a very low overall consolidity index and moderate diversity ratio. The area of the consolidity region R is also moderate supporting the moderate diversity of calculated consolidity points.

Similar approaches can be applied for the fuzzy pulse response of digital or discrete data control systems expressed by their z-transform transfer functions [16,17].

5. Methodology implementation to fuzzy Routh–Hurwitz stability criterion

The work of Routh and Hurwitz [16] gives a method of indicating the presence and number of unstable roots, but not their value. Consider the general form of characteristic equation expressed by (14). The Routh–Hurwitz stability criterion states that for there to be no roots with positive real parts it is a necessary, but not sufficient, condition that all coefficients in the characteristic equation have the same sign and that none is zero. If the above is satisfied, then the necessary and sufficient condition for stability is either

(a) All the Hurwitz determinants of the polynomial are positive, or alternatively.
(b) All coefficients of the first column of Routh array have the same sign. The number of sign changes indicates the number of unstable roots.

The Hurwitz determinants of (15) can be expressed as follows [16]:

$$D_1 = a_1 \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_2 & a_4 \end{vmatrix} \tag{21}$$

..., etc., such that all parameters are expressed in the Arithmetic fuzzy logic-based representation form. All the above determinant operations are carried out following the fuzzy logic-based algebra operations [29].

Illustrative example 2. Let us check the stability of the closed-loop control system of Illustrative example 2, where the open-loop transfer function is expressed in (17), and is having a unity feedback. The closed-loop characteristic function can be expressed as

$$s \cdot (s + b_1) \cdot (s^2 + c_1 \cdot s + c_2) + a_0 = 0 \tag{22}$$

or equivalently

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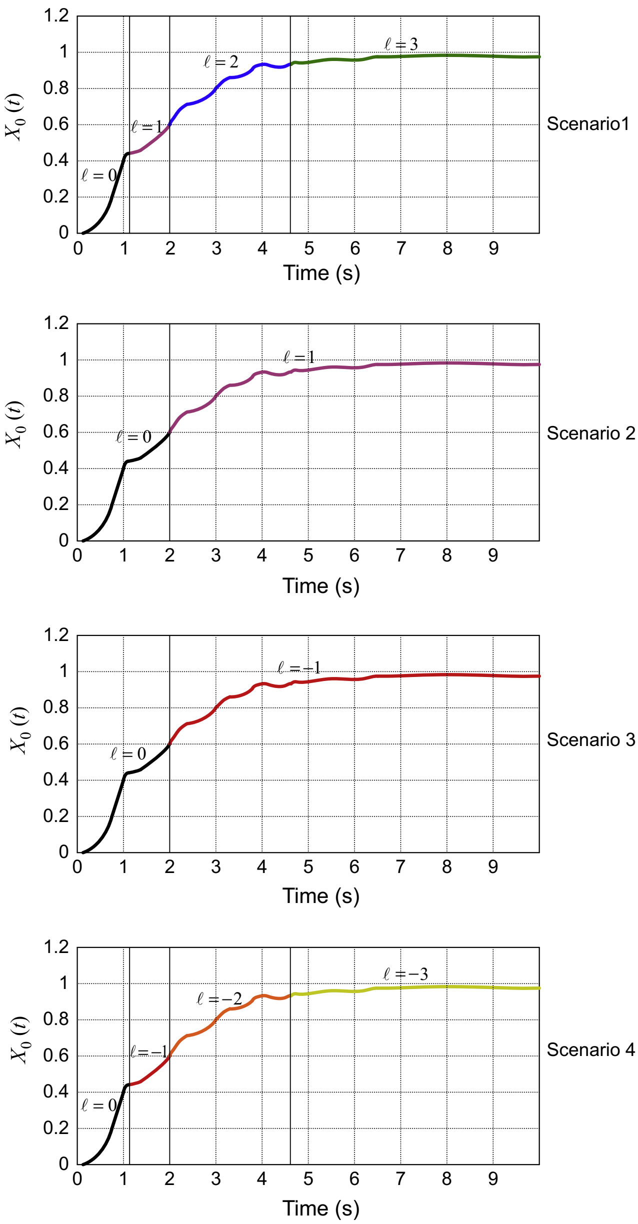

Fig. 6 Fuzzy impulse response using visual representation of Illustrative example 1.

$$s^4 + (c_1 + b_1) \cdot s^3 + (c_2 + c_1 \cdot b_1) \cdot s^2 + c_2 \cdot b_1 \cdot s + a_0 = 0 \quad (23)$$

where a_0, b_1, c_1 and c_2 are fuzzy parameters following the scenarios given in Table 6. Applying now the Routh–Hurwitz criterion, we have first to test that all the coefficients are present and have the same sign. The second test is to check the fuzzy determinants D_1, D_2, and D_3 for different scenarios.

The Hurwitz determinants for this numerical example can be expressed as

$$D_1 = c_2 \cdot b_1 \quad (24)$$

$$D_2 = \begin{vmatrix} c_2 \cdot b_1 & c_1 + b_1 \\ a_1 & c_2 + c_1 \cdot b_1 \end{vmatrix} \quad (25)$$

Table 4 System impulse response and corresponding fuzzy levels of Illustrative example 1.

Ser.	Time (s)	$X_0(t)$	Selected fuzzy levels scenarios			
			I	II	III	IV
1	0.0	0.00000	0	0	0	0
2	1.0	0.38820	1	0	0	−1
3	2.0	0.62573	1	0	0	−1
4	3.0	0.77107	2	1	−1	−2
5	4.0	0.85999	2	1	−1	−2
6	5.0	0.91440	3	1	−1	−3
7	6.0	0.94768	3	1	−1	−3
8	7.0	0.96803	3	1	−1	−3
9	8.0	0.98047	3	1	−1	−3
10	9.0	0.98808	3	1	−1	−3
11	10.0	0.99273	3	1	−1	−3

and

$$D_3 = \begin{vmatrix} c_2 \cdot b_1 & c_1 + b_1 & 0 \\ a_1 & c_2 + c_1 \cdot b_1 & 1 \\ 0 & c_2 \cdot b_1 & c_1 + b_1 \end{vmatrix} \quad (26)$$

The numerical results are shown for example in Table 6.

The Hurwitz determinants of the polynomial are all positive with various level of fuzziness. The fuzzy levels of determinants describe the level of uncertainty in the results. For different scenarios, we will have same fuzzy results with opposite signs of the fuzziness of the determinant of scenarios corresponding with opposite input fuzzy data. This indicates the inverse property of the Arithmetic fuzzy logic-based approach. Similar approach can be applied for solving Jury stability criterion of fuzzy discrete control systems expressed by the z-transform characteristic equation of the sampled data system [16,17].

The consolidity pattern of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{I+S}|$ is shown in Fig. 7. The solution pattern reveals high unconsolidated distribution of the results, indicating ***high susceptibility*** of the Routh–Hurwitz stability criterion for change versus any system and input parameters changes effect. Based on consolidity chart of Fig. 7, it can be seen that the control system is ***unconsolidated*** of class "U".

The analysis of the consolidity chart of the Routh–Hurwitz stability criterion of Illustrative example 2 sketched in Fig. 7 can be summarized as follows:

Symbol	Meaning	Results
R	Shape type	Elliptical
S	Slope	74.54°, or $\tan^{-1}$ (3.6149)
$C = (x, y)$	Centroid	(1.9241, 7.0886)
A	Area of shape	14.7669 pu^2
l_1	Length of major diagonal	6.1772 pu
l_2	Length of minor diagonal	3.0379 pu
l_2/l_1	Diversity ratio	0.4918

The results reveal that the consolidity region has both moderate overall consolidity index and diversity ratio. The area of the consolidity region R is moderate supporting the moderate diversity of calculated consolidity.

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Table 5 Definition of positive and negative color sample scales used in sketching Fig. 6.

Ser.	Color	Color code	RBG color index	Excel color index	Corresponding fuzzy level	Type
1		Violet (lavender)	(255, 0, 255)	7	+6	Positive colors
2		Blue	(0, 102, 204)	5	+5	
3		Green	(51, 153, 102)	50	+4	
4		Violet (lavender light)	(204, 153, 255)	39	+3	
5		Blue light	(153, 204, 255)	37	+2	
6		Green light	(204, 255, 204)	35	+1	
7		Black	(0, 0, 0)	2	0	
8		Yellow light	(255, 255, 204)	19	−1	Negative colors
9		Orange light	(255, 153, 0)	45	−2	
10		Red light	(255, 153, 204)	38	−3	
11		Yellow	(255, 255, 0)	27	−4	
12		Orange	(255, 102, 0)	46	−5	
13		Red	(255, 0, 0)	3	−6	

Table 6 Results of Routh–Hurwitz fuzzy determinants of Illustrative example 2.

Ser.	Aspect	Value	Fuzzy level scenarios								
			I	II	III	IV	V	VI	VII	VIII	IX
1	a_0	12.5	1	3	2	4	4	5	3	−4	−1
2	b_1	0.5	3	1	4	1	2	1	−6	−3	−3
3	c_1	1.0	1	3	3	3	2	2	1	−1	−2
4	c_2	25.0	1	2	1	2	3	3	3	5	−2
5	$\|D_1\|$	12.5	4	3	5	3	5	4	3	2	−5
6	$\|D_2\|$	300.0	5	5	6	5	8	7	0	8	−7
7	$\|D_3\|$	443.75	6	8	8	8	10	9	9	7	−9
Consolidity index $F_{O/(I+S)}$ [a]			5.7855	2.8442	5.6727	2.4726	2.8328	2.2539	1.9153	3.7299	4.9909

[a] Average value of consolidity index $F_{O/(I+S)} = 3.6149$.

6. Methodology implementation to control systems fuzzy controllability and observability

A system is said to be ***controllable*** if a control vector $u(t)$ exists that will transfer the system from any initial state $x(t_o)$ to some final state $x(t)$ in a finite time interval.

A system is said to be ***observable*** if at time t_0, the system state $x(t_0)$ can be exactly determined from observation of the output $y(t)$ over a finite time interval.

If the system is described by

$$\begin{aligned} \dot{x} &= A \cdot x + B \cdot u \\ y &= C \cdot x + D \cdot u \end{aligned} \tag{27}$$

then a sufficient condition for complete state controllability is that the $n \times n$ matrix [16]:

$$M = [B : A \cdot B : \ldots : A^{n-1} \cdot B] \tag{28}$$

contains n linearly independent row or column vectors, i.e. is of rank n (that is, the matrix is non-singular, and the determinant is non-zero). Eq. (28) designates the ***Controllability matrix.***

The system described by (28) is completely observable if the $n \times n$ matrix is denoted as the ***Observability matrix*** [16]:

$$N = [C^T : A^T \cdot C^T : \cdots : (A^T)^{n-1} \cdot C^T] \tag{29}$$

where all the system coefficients are expressed in the Arithmetic fuzzy logic-based representation form.

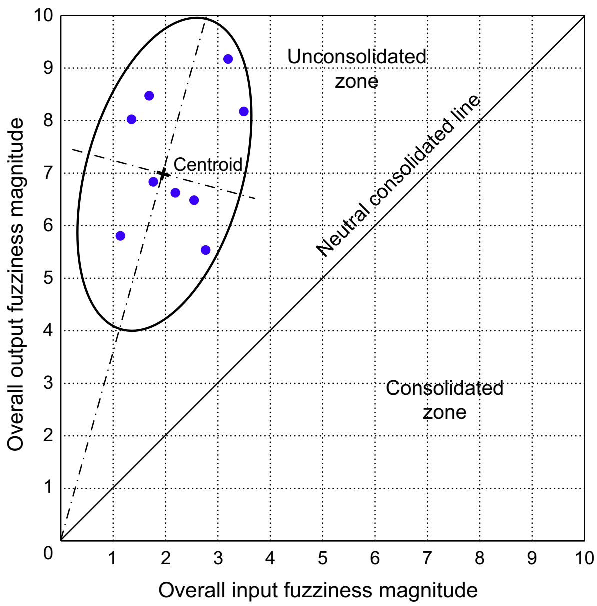

Fig. 7 Consolidity chart of the Routh–Hurwitz stability criterion of Illustrative example 2.

Illustrative example 3. Consider the state space representation of the closed-loop system of Illustrative example 3:

$$\begin{bmatrix} x_1^\bullet \\ x_2^\bullet \\ x_3^\bullet \\ x_4^\bullet \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix} u \tag{30}$$

and

$$y = [0 \quad 0 \quad 0 \quad \beta] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{31}$$

such that $a_1 = c_2 \cdot b_1, a_2 = c_2 + c_1 \cdot b_1, a_3 = c_1 + b_1$ and α, β are fuzzy parameters, where $\alpha = 2$ and $\beta = 1$.

Let us form the Controllability matrix M as

$$M = [B : A \cdot B : A^2 \cdot B : A^3 \cdot B]$$
$$= \alpha^4 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -a_3 \\ 0 & 1 & -a_3 & -a_2 + a_3^2 \\ 1 & -a_3 & -a_2 + a_3^2 & -a_1 + 2 \cdot a_2 \cdot a_3 - a_3^3 \end{bmatrix} \tag{32}$$

This yields the determinant of $M = \alpha^4 \neq 0$, with fuzzy level $= 4 \cdot \ell_\alpha$. Thus the system is controllable. The fuzzy levels of various scenarios are illustrated in Table 7. The results indicate that for this example the fuzzy level of the controllability matrix determinant is a function only of α^4 and not related to other system parameters fuzziness.

Applying the Observability Matrix criterion, we have

$$N = [C^T : A^T \cdot C^T : (A^T)^2 \cdot C^T : (A^T)^3 \cdot C^T]$$
$$= \beta^4 \begin{bmatrix} 0 & -a_0 & a_0 a_3 & a_0 \cdot a_2 - a_0 \cdot a_3^2 \\ 0 & -a_1 & -a_0 + a_1 \cdot a_3 & a_0 \cdot a_3 + a_1 \cdot a_2 - a_1 \cdot a_3^2 \\ 0 & -a_2 & -a_1 + a_2 \cdot a_3 & -a_0 + a_1 \cdot a_3 + a_2^2 - a_2 \cdot a_3^2 \\ 1 & -a_3 & -a_2 + a_3^2 & -a_1 + 2a_2 \cdot a_3 - a_3^3 \end{bmatrix} \tag{33}$$

The determinant of $N \neq 0$, thus the system is also observable. The associated fuzzy levels of various scenarios are also given in Table 7. It follows from the example that the fuzzy level of the Observability matrix determinant is a function of β^4 and also related to the system parameter fuzziness. The results indicate the existence of a relatively high fuzziness of the Observability Matrix due to the fuzziness of the system's parameters.

The consolidity pattern of the controllability and observability problem of Illustrative example 3 is described by plotting the output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{I+S}|$ as shown in Figs. 8 and 9. Both charts reveal moderately unconsolidated distribution of the results, indicating relatively ***medium susceptibility*** of the both conditions for change versus any system and input parameters changes effect. Based on consolidity charts of Figs. 8 and 9, it can be seen that the control system performance is ***unconsolidated*** of class "U" for both controllability and observability.

Table 7 Consolidity results controllability and observability matrix criteria of Illustrative example 3.

Ser.	Aspect	Value	Fuzzy level scenarios								
			I	II	III	IV	V	VI	VII	VIII	IX
1	a_0	12.5	−1	6	3	5	8	5	3	6	3
2	a_1	12.5	2	5	7	4	7	3	9	3	10
3	a_2	25.0	3	2	4	2	3	2	5	1	5
4	a_3	1.5	−2	4	2	2	3	4	2	3	2
5	α	2.0	−1	−2	2	2	2	1	2	−1	2
6	β	1.0	2	−4	1	−3	−4	−1	1	−1	1
7	$\lvert M \rvert$	16.0	−4	−8	8	8	8	4	8	−4	8
Consolidity index value $F_{O/(I+S)}$ [a]			1.9765	3.9529	2.7769	3.6721	2.3579	2.0614	2.3093	2.6047	2.3014
8	$\lvert N \rvert$	-58.085×10^3	9	−8	9	−6	−7	4	8	3	8
Consolidity index value $F_{O/(I+S)}$ [b]			4.5515	4.1228	3.0586	2.7455	1.9272	1.8919	2.4194	2.0869	2.3749

[a] Average value of consolidity index $F_{O/(I+S)} = 2.6681$.
[b] Average value of consolidity index $F_{O/(I+S)} = 2.7976$.

The analysis of the consolidity chart of the fuzzy system ***controllability*** results of Illustrative example 3 sketched in Fig. 8 can be summarized as follows:

Symbol	Meaning	Results
R	Shape type	Elliptical
S	Slope	69.45° or $\tan^{-1}$ (2.6681)
$C = (x, y)$	Centroid	(2.3291,6.3797)
A	Area of shape	18.9713 pu^2
l_1	Length of major diagonal	7.5949 pu
l_2	Length of minor diagonal	3.1392 pu
l_2/l_1	Diversity ratio	0.4133

The results show that the consolidity region has a moderate overall consolidity index and relatively high diversity ratio. The area of the consolidity region is very high supporting the high diversity of calculated consolidity points.

As for the consolidity chart of the fuzzy system ***observability*** results of Illustrative example 3 delineated in Fig. 9, the analysis can be summarized as follows:

Symbol	Meaning	Results
R	Shape type	Elliptical
S	Slope	70.33° or $\tan^{-1}$ (2.7976)
$C = (x, y)$	Centroid	(2.4304,6.5822)
A	Area of shape	13.7414 pu^2
l_1	Length of major diagonal	6.9873 pu
l_2	Length of minor diagonal	2.3291 pu
l_2/l_1	Diversity ratio	0.3333

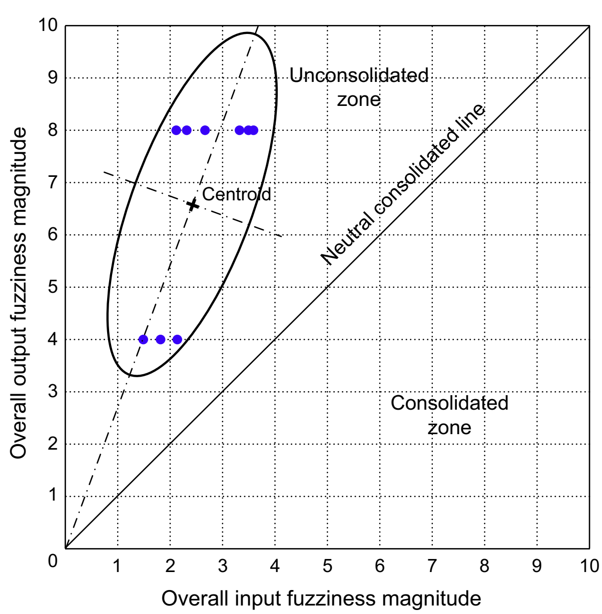

Fig. 8 Consolidity pattern of the fuzzy system ***controllability*** results of Illustrative example 3.

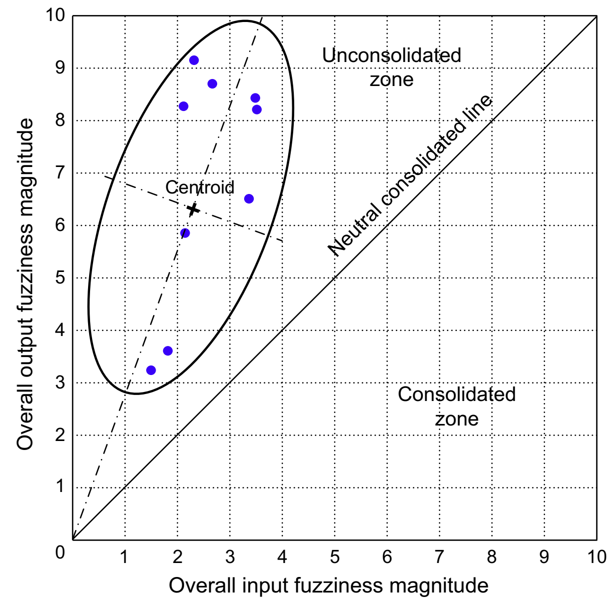

Fig. 9 Consolidity pattern of the fuzzy system ***observability*** results of Illustrative example 3.

The results show that the consolidity region has a moderate overall consolidity index and low diversity ratio. The area of the consolidity region R is high supporting the spread of calculated consolidity points.

Similar approach can be applied for testing the fuzzy controllability and observability of digital control systems expressed by their discrete time state equations [16,17].

7. Methodology implementation to modeling and design of inverted pendulum stabilization

In this section, the suggested approach is implemented for the fuzzy modeling and stabilization of the inverted pendulum system (to be referred to as **Illustrative example 4**) as shown in Fig. 10. The inverted pendulum problem is an example of producing a stable closed-loop control system from an unstable plant. For this system, it is possible to design a controller using the pole placement techniques [16].

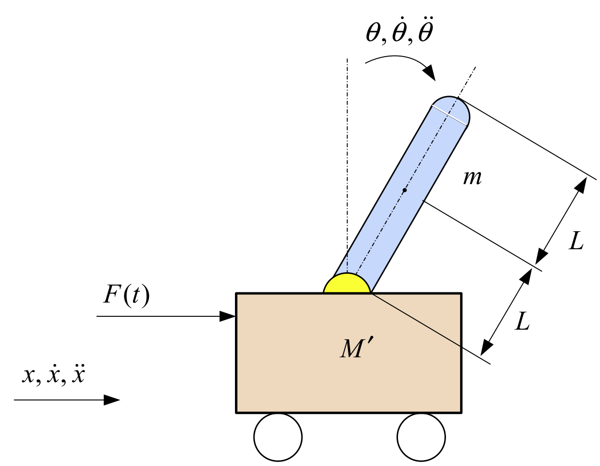

Fig. 10 Sketch showing main parameters of the inverted pendulum of **Illustrative example 4**.

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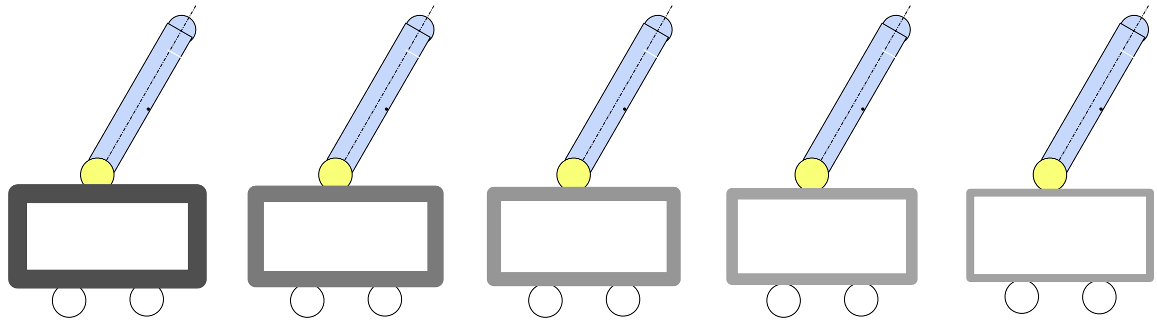

Fig. 11 Various selected design prototypes of inverted pendulum of different manufacturing materials of **Illustrative example 4** (Case 1: $M' = 8$, Case 2: $M' = 4$, Case 3: $M' = 2$, Case 4: $M' = 1$, Case 5: $M' = 0.6$).

Table 8 Consolidity analysis of inverted pendulum of Case 1: $M' = 8$ of **Illustrative example 4**.

Type	Parameter	Value	Corresponding fuzzy level values								
			I	II	III	IV	V	VI	VII	VIII	IX
Input	L	1	2	2	4	3	2	3	4	3	4
	m	0.5	3	4	3	5	2	6	5	3	7
	a_{21}	9.81	−2	−2	−4	−3	−2	−3	−4	−3	−4
	a_{41}	−3.27	−3	−4	−4	−5	−2	−6	−5	−3	−7
	b_2	−0.6667	−2	−2	−4	−3	−2	−3	−4	−3	−4
	b_4	0.8889	0	0	0	0	0	0	0	0	0
Output	$\|M\|$	0.007	−8	−8	−16	−11	−8	−11	−15	−12	−15
	k_1	−212.0	1	1	3	2	1	2	3	2	3
	k_2	282.7	1	0	1	1	1	1	1	1	1
	k_3	−412.7	2	2	4	3	2	3	4	3	3
	k_4	−1240.3	2	2	4	3	2	3	4	3	3
Consolidity index value $F_{O/(I+S)}$ [a]			3.4779	3.0220	4.4727	3.3121	4.0858	3.0220	3.7585	4.0858	3.2348

[a] Average value of consolidity index $F_{O/(I+S)} = 3.6080$.

In the figure, m is the mass of pendulum, L denotes the half-length of the pendulum and M' is the mass of the trolley. The parameter $F(t)$ indicates the applied force to the trolley in the x-direction.

It is assumed that θ is small and the second-order terms $(\dot{\theta}^2)$ can be neglected, then we can define the state variables of the inverted pendulum system as $(g = 9.81)$

$$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = x \quad \text{and} \quad x_4 = \dot{x} \tag{34}$$

and the control variable is

$$u = F(t) \tag{35}$$

From (34) and (35), the state equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} u \tag{36}$$

where

$$\begin{aligned} a_{21} &= \frac{3 \cdot g \cdot (M' + m)}{L \cdot [4 \cdot (M' + m) - 3 \cdot m]} \\ a_{41} &= \frac{-3g \cdot m}{4 \cdot (M' + m) - 3 \cdot m} \\ b_2 &= \frac{-3}{L \cdot [4 \cdot (M' + m) - 3 \cdot m]} \\ b_4 &= \left(\frac{1}{M' + m}\right) \cdot \left\{1 + \frac{3 \cdot m}{4 \cdot (M' + m) - 3 \cdot m}\right\} \end{aligned} \tag{37}$$

$$M = \begin{bmatrix} 0 & b_2 & 0 & a_{21} \cdot b_2 \\ b_2 & 0 & a_{21} \cdot b_2 & 0 \\ 0 & b_4 & 0 & a_{41} \cdot b_2 \\ b_4 & 0 & a_{41} \cdot b_2 & 0 \end{bmatrix} \tag{38}$$

Data for simulation are represented in Table 8 for different selected scenarios. The output equation is

$$y = C \cdot x \tag{39}$$

where C is the identity matrix. For a regulator, with a scalar control variable and gain vector K, we have

$$u = -K \cdot x \tag{40}$$

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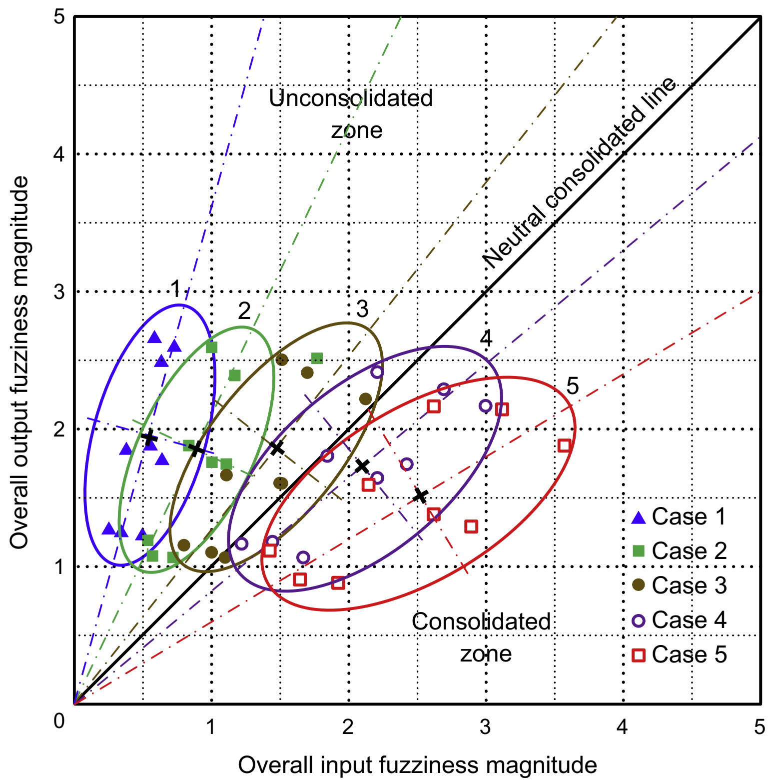

Fig. 12 Consolidity patterns of the fuzzy inverted pendulum designs of **Illustrative example 4**.

The elements of K can be obtained by selecting a set of desired closed-loop poles by the Ackermann's formula [16] for system stabilization through the pole placement technique. Let

$$K = [0 \quad 0 \quad 0 \quad \cdots \quad 1] \cdot M^{-1} \cdot \phi(A) \tag{41}$$

where M is the ***Controllability matrix*** and

$$\phi(A) = A^n + \alpha_{n-1} \cdot A^{n-1} + \cdots + \alpha_1 \cdot A + \alpha_0 \cdot I \tag{42}$$

where A is the system matrix and α_i represent the coefficients of the desired closed-loop characteristic equation.

If the required closed-loop poles are $s = -2 \pm j2$ for the pendulum, and $s = -4 \pm j4$ for the trolley, then the closed-loop characteristic equation becomes

$$s^4 + 12s^3 + 72s^2 + 192s + 256 = 0 \tag{43}$$

The algebraic form of the fuzzy gain result can be expressed as follows: let

$$[\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4] = [0 \quad 0 \quad 0 \quad 1] M^{-1} \tag{44}$$

then we can attain by simple matrix operation the following fuzzy values of the gain vector K:

$$k_1 = \beta_1 \cdot (\alpha_o + \alpha_2 \cdot a_{21} + \alpha_4 \cdot a_{21}^2) + \beta_2(\alpha_2 \cdot a_{41} + \alpha_4 \cdot a_{21} \cdot a_{41}) \tag{45}$$

$$k_2 = \beta_1 \cdot (\alpha_1 + \alpha_3 \cdot a_{21}) + \beta_3 \cdot \alpha_3 \cdot a_{41} \tag{46}$$

$$k_3 = \beta_3 \cdot \alpha_0 \tag{47}$$

and

$$k_4 = \beta_3 \cdot \alpha_1 \tag{48}$$

For design purposes, various prototypes of the inverted pendulum can be selected with different relative trolley and pendulum masses of different manufacturing materials and with gain vector K satisfying the same stabilized characteristics

Symbol	Results of various designs of M'				
	8.0	4.0	2.0	1.0	0.6
R	Elliptical	Elliptical	Elliptical	Elliptical	Elliptical
S	74.51° or $\tan^{-1}$ (3.6080)	63.96° or $\tan^{-1}$ (2.0467)	51.42° or $\tan^{-1}$ (1.2534)	39.60° or $\tan^{-1}$ (0.8274)	32.23° or $\tan^{-1}$ (0.6305)
$C = (x, y)$	(0.5250, 1.9250)	(0.8500, 1.8400)	(1.4500, 1.8450)	(2.0750, 1.5000)	(2.5000, 1.5000)
$A(\text{pu}^2)$	1.2500	1.2750	1.6500	2.2750	2.5000
l_1(pu)	1.9500	1.9500	2.2000	2.4000	2.5500
l_2	0.8000	0.8500	0.9500	1.2000	1.2500
l_2/l_1	0.4103	0.4359	0.4318	0.5000	0.4902

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function of (34). The most desired robust design can be attained that achieves the best consolidity performance. In this respect, five different designs of trolley car masses M' of 8, 4, 2, 1 and 0.6 respectively are selected as shown in Fig. 11.

For each design, the consolidity analysis is applied and the result is shown in Table 8 for Case 1 of $M' = 8$ and sketched all the three cases in Fig. 12. From this figure, it appears that the consolidity pattern of the inverted pendulum improves with the reduction of the trolley car mass M' and the best consolidity performance is obtained for the fifth design case with $M' = 0.6$. The analysis of the consolidity chart of the fuzzy inverted pendulum of various selected designs of **illustrative example 4** delineated in Fig. 12 can be summarized as follows:

It follows from the above table that reducing the trolley weight the system overall consolidity is improved but is accompanied with increase of the consolidity region areas. In general, the diversity ratio is high indicating high diversity of calculated consolidity points. The areas of the regions are in general very small compared to all previous illustrative examples ($A > 13$). Such charts of Fig. 12 clearly demonstrate the effectiveness of the proposed approach as a tool for the analysis and design of automatic control systems.

8. Conclusions

A new approach for the fuzzy automatic control systems' modeling and analysis using the consolidity theory was presented. Key implementations issues in fuzzy automatic control and dynamical systems were addressed in a very smooth and systematic way. These issues covered system's fuzzy impulse response, system's stability using Routh–Hurwitz criterion, system's Controllability and Observability, and the stabilization of inverted pendulum through the pole placement technique. Illustrative examples of fourth-order systems were solved to demonstrate the effectiveness and applicability of the new technique. The approach is also suitable for higher dimensional automatic control and dynamical systems as it is based mainly on matrix formulations.

Implementation procedures were elaborated for the consolidity analysis of existing control systems and the design of new ones. Systems comparisons based on such implementation consolidity results were discussed based on the values of the average, weighted or worst consolidity scores, or the comparison of the plotted regions (patterns) of fuzziness behavior in the sketched consolidity chart. It is shown that the regions comparison in consolidity chart is based on type of consolidity region shape, slope or angle in degrees of the centerline of the geometric shape, the centroid of the geometric shape, area of the geometric shape, length of principal diagonals of the shape, and the diversity ratio of consolidity points for each region.

The suggested approach could open the door for more general modeling, analysis and design of both *continuous* and *digital* data automatic control systems. Moreover, it provides an effective tool for designing new control systems that could withstand future changes due to any system or input parameters changes effects on and above normal systems operations or set points. Examples of some foreseen extensions are as follows:

(i) Design of Fuzzy P, PI, PD, and PID control systems.
(ii) State-space methods for fuzzy automatic control system analysis and design.
(iii) Design of fuzzy state observers and estimators for closed-loop systems.
(iv) Optimal and robust fuzzy control of multivariate systems.
(v) Other problems such as multivariate fuzzy Kalman state estimation, fuzzy linear quadratic regulators, and fuzzy Lyapunov stability criterion.

It was shown that the presented consolidity methodology is open in its application to wide classes of systems. Even for the system that thought not to be fuzzy, we can still imagine that these systems are operating in a fully fuzzy environment and perform typically the same consolidity testing. Its needless to say that all the present physical systems in our daily life are subject to continuous wearing and deterioration that make them gradually changing, and thus will behave later equivalently as if they are operating in a fuzzy environment. This makes the presented ***consolidity chart*** approach *generally* extendable to wider spectrum of real-life applications beside that given in the automatic control fields. Examples of these fields are geology, archeology, life sciences, ecology, environmental science, engineering, materials, medicine, biology, sociology, humanities, and many other important fields.

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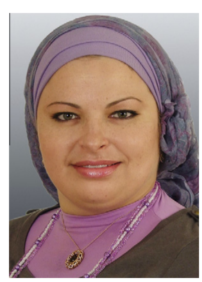

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